

CLASSIFICATIONS OF LINEAR OPERATORS PRESERVING ELLIPTIC, POSITIVE AND NON-NEGATIVE POLYNOMIALS

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ABSTRACT. We characterize all linear operators on finite or infinite-dimensional spaces of univariate real polynomials preserving the sets of elliptic, positive, and non-negative polynomials, respectively. This is done by means of Fischer-Fock dualities, Hankel forms, and convolutions with non-negative measures. We also establish higher-dimensional analogs of these results. In particular, our classification theorems solve the questions raised in [9] originating from entire function theory and the literature pertaining to Hilbert's 17th problem.

1. INTRODUCTION

The dynamics of zero sets of polynomials and transcendental entire functions under linear transformations is a central topic in geometric function theory with applications ranging from statistical mechanics and probability theory to combinatorics, analytic number theory, and matrix theory; see, e.g., [3]–[8] and the references therein. Describing linear operators that preserve non-vanishing properties is a basic question that often turns out to be quite difficult. For instance, the problem – going back to Laguerre and Pólya-Schur [19] – of characterizing linear operators preserving univariate real polynomials with *all real zeros* remained unsolved until very recently [4]. Just as fundamental are the problems – originating from entire function theory [11, 12, 13, 17, 20] and the literature pertaining to Hilbert's 17th problem [16, 21, 26] – of classifying linear operators preserving the set of univariate real polynomials with *no real zeros* or the closely related sets of positive, respectively non-negative polynomials. Let us formulate these problems explicitly.

Given $d \in \mathbb{N}_0$, where \mathbb{N}_0 is the additive semigroup of non-negative integers, let $\mathbb{R}_d[x] = \{p \in \mathbb{R}[x] : \deg(p) \leq d\}$ and set

$$POS = \{p \in \mathbb{R}[x] : p(x) > 0, x \in \mathbb{R}\}, \quad POS_d = POS \cap \mathbb{R}_d[x],$$

$$SOS = \left\{ \sum_{i=1}^r p_i(x)^2 : p_i \in \mathbb{R}[x], 1 \leq i \leq r < \infty \right\}, \quad SOS_d = SOS \cap \mathbb{R}_d[x],$$

$$ELL = \{p \in \mathbb{R}[x] : p(x) \neq 0, x \in \mathbb{R}\}, \quad ELL_d = ELL \cap \mathbb{R}_d[x].$$

It is well-known that $SOS = \overline{POS} := \{p \in \mathbb{R}[x] : p(x) \geq 0, x \in \mathbb{R}\}$, $ELL = POS \cup (-POS)$, and $POS = \cup_{a>0} (SOS + a)$. Polynomials in POS, SOS, ELL are called *positive*, *non-negative* or *sos* (sums of squares), and *elliptic*, respectively. Let \mathfrak{L}_d be the set of all linear operators $T : \mathbb{R}_d[x] \rightarrow \mathbb{R}[x]$ and \mathfrak{L} be the semigroup of all linear operators $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$. For $V \in \{POS, SOS, ELL\}$ and $d \in \mathbb{N}_0$ define

$$\mathfrak{L}(V) = \{T \in \mathfrak{L} : T(V) \subseteq V\}, \quad \mathfrak{L}_d(V) = \{T \in \mathfrak{L}_d : T(V_d) \subseteq V\}.$$

Problem 1. Characterize $\mathfrak{L}_d(POS)$, $\mathfrak{L}_d(SOS)$, $\mathfrak{L}_d(ELL)$ for each even $d \in \mathbb{N}_0$.

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Problem 2. *Characterize the semigroups $\mathfrak{L}(\text{POS})$, $\mathfrak{L}(\text{SOS})$, $\mathfrak{L}(\text{ELL})$.*

In physics it is useful to distinguish between “soft” and “hard” theorems asserting the non-vanishing of partition functions in certain regions.¹ By analogy with this terminology and the one used in the classification of linear preservers of real-rooted polynomials [3, 4], one may say that results pertaining to Problem 1 are “hard” or “algebraic” (bounded degree) while those for Problem 2 are “soft” or “transcendental” (unbounded degree). Special cases of Problems 1–2 were recently studied in [9], where one can also find a list of partial results known so far and corrections to some erroneous claims from the existing literature.

In this paper we give complete solutions to Problems 1–2. In particular, this supersedes [9] whose main results are now immediate corollaries of the classification theorems in §2. In §3 we answer multivariate versions of Problems 1–2 while in §4 we establish some related results and discuss further directions.

One of the tools that we use both in the univariate and the multivariate case is the following inner product structure on polynomial spaces. Recall (cf., e.g., [6] and the references therein) that the *Fischer-Fock* or *Bargmann-Segal space* \mathcal{F}_n is the Hilbert space of holomorphic functions f on \mathbb{C}^n such that

$$\|f\|^2 := \pi^{-n} \int_{\mathbb{C}^n} |f(\mathbf{z})|^2 e^{-|\mathbf{z}|^2} dV(\mathbf{z}) < \infty,$$

where $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, $|\mathbf{z}|^2 = \sum_{i=1}^n |z_i|^2$, and $dV(\mathbf{z})$ is the volume element (Lebesgue measure) in \mathbb{C}^n . The inner product in \mathcal{F}_n is given by

$$\begin{aligned} \langle f, g \rangle &= \pi^{-n} \int_{\mathbb{C}^n} f(\mathbf{z}) \overline{g(\mathbf{z})} e^{-|\mathbf{z}|^2} dV(\mathbf{z}) \\ &= f(\partial/\partial z_1, \dots, \partial/\partial z_n) g(z_1, \dots, z_n) \Big|_{z_1=\dots=z_n=0} = \sum_{\alpha \in \mathbb{N}_0^n} \alpha! a_\alpha \bar{b}_\alpha. \end{aligned}$$

Here $\alpha! = \alpha_1! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ while $\sum_\alpha a_\alpha \mathbf{z}^\alpha$, respectively $\sum_\alpha b_\alpha \mathbf{z}^\alpha$, is the Taylor expansion of f , respectively g , where $\mathbf{z}^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Clearly, the space of all polynomials in n variables with real coefficients $\mathbb{R}[z_1, \dots, z_n]$ is contained in \mathcal{F}_n and thus inherits a natural inner product structure. Given a cone $K \subseteq \mathbb{R}[z_1, \dots, z_n]$ we let $K^* \subseteq \mathbb{R}[z_1, \dots, z_n]$ be its *dual* cone, i.e.,

$$K^* = \{f \in \mathbb{R}[z_1, \dots, z_n] : \langle f, g \rangle \geq 0, g \in K\}.$$

The Fischer-Fock space \mathcal{F}_n was used by Dirac to define second quantization and its inner product has since been rediscovered in various contexts, for instance in number theory – where the corresponding norm is known as the Bombieri norm – and the theory of real homogeneous polynomials [26]. In \mathcal{D} -module theory and microlocal Fourier analysis one usually works with the inner product on \mathcal{F}_n defined by $(f(\mathbf{z}), g(\mathbf{z})) = \langle f(i\mathbf{z}), g(i\mathbf{z}) \rangle$. For these and further properties of \mathcal{F}_n (such as its Bergman-Szegő-Aronszajn reproducing kernel) see, e.g., the references in [6].

Another main ingredient in our characterization theorems is the theory of positive definite kernels, Hankel forms, and moment problems in one or several variables. There is a vast literature on these topics ranging from classical works on univariate and multivariate moment problems [1, 10, 14, 15, 18, 28, 29] to recent contributions [2, 16, 22, 23, 24, 26] and we refer to these for additional details. As usual, a $k \times k$ real symmetric matrix A is said to be positive semidefinite (respectively, positive definite) if $\mathbf{x}^t A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^k$ (respectively, $\mathbf{x}^t A \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^k \setminus \{0\}$).

¹This dichotomy stems from “soft-core” vs. “hard-core” pair interactions in lattice-gas models and does not refer to the level of difficulty in proving such theorems but rather to the fact that in some sense “soft” theorems are constraint-free while “hard” theorems involve constraints of various kinds such as the maximum degree of a graph; see, e.g., [3] and the references therein.

Let \mathfrak{M}_1^+ denote the class of non-negative measures μ on \mathbb{R} with all finite moments, that is, $\int |t^m| d\mu(t) < \infty$, $m \in \mathbb{N}_0$. Hamburger's theorem [14] asserts that a sequence of real numbers $\{a_m\}_{m \in \mathbb{N}_0}$ is the moment sequence of a measure $\mu \in \mathfrak{M}_1^+$, i.e.,

$$a_m = \int t^m d\mu(t), \quad m \in \mathbb{N}_0,$$

if and only if the Hankel matrix $(a_{i+j})_{i,j=0}^m$ is positive semidefinite for any $m \in \mathbb{N}_0$. If such a μ is unique the corresponding moment problem is said to be determined.

Haviland's theorem [15] is a multivariate analog of Hamburger's theorem that may be stated as follows (cf., e.g., [2, Chap. 6]). Let $n \in \mathbb{N}$ and \mathfrak{M}_n^+ be the class of non-negative measures μ on \mathbb{R}^n with all finite moments, that is, $\int |\mathbf{t}^\alpha| d\mu(\mathbf{t}) < \infty$, $\alpha \in \mathbb{N}_0^n$, where $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $\mathbf{t}^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. A sequence of real numbers $\{a_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ satisfies

$$a_\alpha = \int \mathbf{t}^\alpha d\mu(\mathbf{t}), \quad \alpha \in \mathbb{N}_0^n,$$

for some $\mu \in \mathfrak{M}_n^+$ if and only if the following holds: the real-valued linear functional L on $\mathbb{R}[x_1, \dots, x_n]$ determined by $L(\mathbf{x}^\alpha) = a_\alpha$, $\alpha \in \mathbb{N}_0^n$, satisfies $L(f) \geq 0$ for any non-negative polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ (i.e., $f(\mathbf{x}) \geq 0$, $\mathbf{x} \in \mathbb{R}^n$).

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2. UNIVARIATE POLYNOMIALS

2.1. Preliminaries. For convenience, we recall some elementary properties of the classes of linear operators under consideration. If $T \in \mathfrak{L}_d$ and $T(1) \in ELL$ then the following are equivalent: (i) $T \in \mathfrak{L}_d(ELL)$; (ii) $T \in \mathfrak{L}_d(POS)$ or $-T \in \mathfrak{L}_d(POS)$; (iii) $T \in \mathfrak{L}_d(SOS)$ or $-T \in \mathfrak{L}_d(SOS)$. Similarly, if $T \in \mathfrak{L}$ and $T(1) \in ELL$ then the following are equivalent: (i') $T \in \mathfrak{L}(ELL)$; (ii') $T \in \mathfrak{L}(POS)$ or $-T \in \mathfrak{L}(POS)$; (iii') $T \in \mathfrak{L}(SOS)$ or $-T \in \mathfrak{L}(SOS)$. If one drops the assumption $T(1) \in ELL$ then (i) \Leftrightarrow (ii) \Rightarrow (iii) and (i') \Leftrightarrow (ii') \Rightarrow (iii'). Furthermore, if $T \in \mathfrak{L}(SOS)$ and $T(1) \equiv 0$ then $T \equiv 0$. These are simple consequences of straightforward homotopy arguments (see, e.g., Remark 4.2 below and [9]).

2.2. Transcendental Characterizations. We start with the “soft” (unbounded degree) case and the three classification questions stated in Problem 2. Recall that any linear operator $T \in \mathfrak{L}$ may be uniquely represented as a formal power series in $D := \frac{d}{dx}$ with polynomial coefficients, i.e.,

$$T = \sum_{i=0}^{\infty} q_i(x) D^i, \quad q_i \in \mathbb{R}[x], \quad i \in \mathbb{N}_0, \quad (2.1)$$

see, e.g., [6]. Given T as above we associate to it a one-parameter family of linear differential operators with *constant* coefficients $\{T_y\}_{y \in \mathbb{R}}$ defined as follows:

$$T_y = \sum_{i=0}^{\infty} q_i(y) D^i.$$

Note that in general T_y has infinite order and $T_y = T$, $y \in \mathbb{R}$, if T has constant coefficients. For each $y \in \mathbb{R}$ and $m \in \mathbb{N}_0$ we also define a polynomial $p_{y,m} = p_{T_y,m}$ of degree at most m and an $(m+1) \times (m+1)$ Hankel matrix $\mathcal{H}_{y,m} = \mathcal{H}_{T_y,m}$ by

$$p_{y,m}(x) = \sum_{i=0}^m q_i(y) x^i \in \mathbb{R}[x], \quad \mathcal{H}_{y,m} = ((i+j)! q_{i+j}(y))_{i,j=0}^m.$$

Remark 2.1. Clearly, $p_{y,m}(x) \in \mathbb{R}[x, y]$ for each $m \in \mathbb{N}_0$. We call this polynomial in two variables the m -truncated symbol of the operator $T \in \mathfrak{L}$.

Our first theorem characterizes the semigroup $\mathfrak{L}(SOS)$ of linear operators preserving the cone SOS of non-negative/sos polynomials.

Theorem 2.1. *Let $T \in \mathfrak{L}$ be as in (2.1). The following assertions are equivalent:*

- (1) $T \in \mathfrak{L}(SOS)$;
- (2) $T_y \in \mathfrak{L}(SOS)$ for all $y \in \mathbb{R}$;
- (3) $\langle p_{y,2m}, f \rangle \geq 0$ for all $y \in \mathbb{R}$, $m \in \mathbb{N}_0$, $f \in SOS_{2m}$, i.e., $p_{y,m} \in SOS_{2m}^*$;
- (4) $\mathcal{H}_{y,m}$ is positive semidefinite for all $y \in \mathbb{R}$, $m \in \mathbb{N}_0$;
- (5) For any $s \in \mathbb{R}$ there exists $\mu_s \in \mathfrak{M}_1^+$ such that $\int t^i d\mu_s(t) \in \mathbb{R}[s]$, $i \in \mathbb{N}_0$, and $i!q_i(y) = \int t^i d\mu_y(t)$, $y \in \mathbb{R}$, $i \in \mathbb{N}_0$. The family of operators $\{T_y\}_{y \in \mathbb{R}}$ is then given by the convolutions

$$T_y(f)(x) = \int f(t+x) d\mu_y(t), \quad f \in \mathbb{R}[x], y \in \mathbb{R}.$$

Proof. Clearly, (2) \Rightarrow (1). If (1) holds then since SOS is invariant under shift operators e^{aD} , $a \in \mathbb{R}$, one gets $e^{-aD} T e^{aD} \in \mathfrak{L}(SOS)$ hence

$$\sum_{i=0}^{\infty} q_i(x-a) f^{(i)}(x) \geq 0$$

for any $f \in SOS$ and $a, x \in \mathbb{R}$, which proves (2) (note that for each $f \in \mathbb{R}[x]$ the sum in the above inequality is finite). Next, if $a, y \in \mathbb{R}$ and $f \in \mathbb{R}_d[x]$, $d \in \mathbb{N}_0$, then

$$T_y(f)(a) = \sum_{i=0}^d q_i(y) f^{(i)}(a) = \langle p_{y,d}, e^{aD} f \rangle. \quad (2.2)$$

Since $f \in SOS_d \Leftrightarrow e^{aD} f \in SOS_d$, $a \in \mathbb{R}$, identity (2.2) shows that if (2) holds then $\langle p_{y,d}, f \rangle \geq 0$ for all $y \in \mathbb{R}$, $d \in \mathbb{N}_0$, $f \in SOS_d$. In particular, this is true for all even d , which proves (3). Conversely, if (3) holds then (2) follows from (2.2) since any $f \in SOS$ has even degree. Now, for $m \in \mathbb{N}_0$, $y \in \mathbb{R}$, and $g(x) = \sum_{i=0}^m a_i x^i \in \mathbb{R}_m[x]$ one has

$$\langle p_{y,2m}, g^2 \rangle = \sum_{i,j=0}^m (i+j)! q_{i+j}(y) a_i a_j$$

and thus (3) \Leftrightarrow (4) since any $f \in SOS$ is a sos. Finally, by Hamburger's theorem (see §1) the following holds for each $y \in \mathbb{R}$: $\mathcal{H}_{y,m}$ is positive semidefinite for all $m \in \mathbb{N}_0$ if and only if there exists $\mu_y \in \mathfrak{M}_1^+$ such that $i!q_i(y) = \int t^i d\mu_y(t)$, $i \in \mathbb{N}_0$. Since all q_i must be polynomials we conclude that (4) \Leftrightarrow (5), which completes the proof of the theorem. \square

Remark 2.2. By Carleman's sufficient condition for the Hamburger moment problem to be determined [10], if $y \in \mathbb{R}$ is such that $\sum_{i=1}^{\infty} [(2i)! q_{2i}(y)]^{-1/2i} = \infty$ then the corresponding measure $\mu_y \in \mathfrak{M}_1^+$ in Theorem 2.1 (5) is unique. Recent progress on the determinateness of multivariate moment problems has been made in [22, 23, 24].

The following analog of Theorem 2.1 characterizes the semigroup $\mathfrak{L}(POS)$ of linear operators preserving the cone POS of positive polynomials.

Theorem 2.2. *Let $T \in \mathfrak{L}$ be as in (2.1). The following assertions are equivalent:*

- (1) $T \in \mathfrak{L}(POS)$;
- (2) $T_y \in \mathfrak{L}(POS)$ for all $y \in \mathbb{R}$;
- (3) $\langle p_{y,2m}, f \rangle > 0$ for all $y \in \mathbb{R}$, $m \in \mathbb{N}_0$, $f \in POS_{2m}$;
- (4) $\mathcal{H}_{y,m}$ is positive definite for all $y \in \mathbb{R}$, $m \in \mathbb{N}_0$;
- (5) $\det(\mathcal{H}_{y,m}) > 0$ for all $y \in \mathbb{R}$, $m \in \mathbb{N}_0$;

- (6) For any $s \in \mathbb{R}$ there exists $\mu_s \in \mathfrak{M}_1^+$ whose support $\text{supp}(\mu_s)$ is an infinite set such that $\int t^i d\mu_s(t) \in \mathbb{R}[s]$, $i \in \mathbb{N}_0$, and $i!q_i(y) = \int t^i d\mu_y(t)$, $y \in \mathbb{R}$, $i \in \mathbb{N}_0$. The family of operators $\{T_y\}_{y \in \mathbb{R}}$ is then given by the convolutions

$$T_y(f)(x) = \int f(t+x) d\mu_y(t), \quad f \in \mathbb{R}[x], y \in \mathbb{R}.$$

Proof. Recall that $POS = \cup_{a>0}(SOS + a)$ and note that if $T \in \mathfrak{L}(POS)$ then $q_0 \in POS$. Using these observations and the fact that the cones POS , POS_{2m} , $m \in \mathbb{N}_0$, are invariant under shift operators e^{aD} , $a \in \mathbb{R}$, one then gets (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (6) as in the proof of Theorem 2.1 (see, e.g., [28, Theorem 1.2] and Remark 2.3 in [2, Chap. 6] for the extra condition that the supports of the representing measures should be infinite sets, which is not required in Theorem 2.1 (5)). The equivalence (4) \Leftrightarrow (5) is a classical result in matrix theory due to Jacobi and independently to Hurwitz, cf., e.g., [18, 28]. \square

By the remarks in §2.1 Theorem 2.2 also yields a description of the semigroup $\mathfrak{L}(ELL)$ of linear operators preserving the set of elliptic polynomials. Problem 2 is therefore completely solved.

2.3. Algebraic Characterizations. Since there are no positive, non-negative or elliptic polynomials of odd degree, for the “hard” (bounded degree) classification questions stated in Problem 1 it is enough to consider only linear operators $T \in \mathfrak{L}_d$ with $d = 2k$, $k \in \mathbb{N}_0$. It is well-known that any such operator may be viewed as an element of the Weyl algebra $\mathcal{A}_1(\mathbb{R})$ of order at most d (cf., e.g., [6]), that is, a linear ordinary differential operator with polynomial coefficients of the form

$$T = \sum_{i=0}^d q_i(x) D^i, \quad q_i \in \mathbb{R}[x], i \in \mathbb{N}_0. \quad (2.3)$$

By analogy with the “soft” case treated in §2.2 we associate to T a one-parameter family of linear differential operators of order at most d with *constant* coefficients $\{T_y\}_{y \in \mathbb{R}}$, a one-parameter family of polynomials $\{p_{y,d}\}_{y \in \mathbb{R}}$ in $\mathbb{R}_d[x]$, and a one-parameter family of $(d+1) \times (d+1)$ Hankel matrices $\{\mathcal{H}_{y,d}\}_{y \in \mathbb{R}}$ defined as follows:

$$T_y = \sum_{i=0}^d q_i(y) D^i, \quad p_{y,d}(x) = \sum_{i=0}^d q_i(y) x^i \in \mathbb{R}_d[x], \quad \mathcal{H}_{y,d} = ((i+j)!q_{i+j}(y))_{i,j=0}^d.$$

Remark 2.3. Given $d \in \mathbb{N}_0$ it is clear that $p_{y,d} \in \mathbb{R}[x, y]$. This two variable polynomial is called the *algebraic symbol* of the operator $T \in \mathfrak{L}_d$.

The finite degree versions of Theorem 2.1 and Theorem 2.2 given below describe the sets $\mathfrak{L}_d(SOS)$ and $\mathfrak{L}_d(POS)$, respectively. We omit their proofs since they are almost identical to those of Theorems 2.1–2.2.

Theorem 2.3. For $d = 2k$ and $T \in \mathfrak{L}_d$ as in (2.3) the following are equivalent:

- (1) $T \in \mathfrak{L}_d(SOS)$;
- (2) $T_y \in \mathfrak{L}_d(SOS)$ for all $y \in \mathbb{R}$;
- (3) $\langle p_{y,d}, f \rangle \geq 0$ for all $y \in \mathbb{R}$, $f \in SOS_d$, i.e., $p_{y,d} \in SOS_d^*$;
- (4) $\mathcal{H}_{y,d}$ is positive semidefinite for all $y \in \mathbb{R}$.

Remark 2.4. In his work [27] on the truncated Hamburger moment problem M. Riesz showed that the Hankel matrix associated to a given finite sequence of real numbers is positive semidefinite if and only if the sequence satisfies certain equalities and inequalities involving moments of non-negative measures. Using this result one can add a fifth condition to Theorem 2.3 (equivalent to those already stated) which is similar to Theorem 2.1 (5).

Theorem 2.4. *For $d = 2k$ and $T \in \mathfrak{L}_d$ as in (2.3) the following are equivalent:*

- (1) $T \in \mathfrak{L}_d(POS)$;
- (2) $T_y \in \mathfrak{L}_d(POS)$ for all $y \in \mathbb{R}$;
- (3) $\langle p_{y,d}, f \rangle > 0$ for all $y \in \mathbb{R}$, $f \in POS_d$;
- (4) $\mathcal{H}_{y,d}$ is positive definite for all $y \in \mathbb{R}$;
- (5) $\det(\mathcal{H}_{y,m}) > 0$ for all $y \in \mathbb{R}$, $0 \leq m \leq d$.

Remark 2.5. In the case of finite order linear ordinary differential operators with constant coefficients Theorems 2.1–2.2 reduce to classical results of Hurwitz [17] and Remak [25], see also Pólya-Szegő's [20, Chap. VII].

Note that from Theorem 2.4 and the observations in §2.1 one also gets a characterization of the set $\mathfrak{L}_d(ELL)$. Therefore, these results fully solve Problem 1.

3. MULTIVARIATE POLYNOMIALS

We will now establish multivariate analogs of the characterization theorems in §2. Given $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ let

$$\begin{aligned} \mathbb{R}_\alpha[x_1, \dots, x_n] &= \{p \in \mathbb{R}[x_1, \dots, x_n] : \deg_{x_i}(p) \leq \alpha_i, 1 \leq i \leq n\}, \\ POS(n) &= \{p \in \mathbb{R}[x_1, \dots, x_n] : p(x_1, \dots, x_n) > 0, (x_1, \dots, x_n) \in \mathbb{R}^n\}, \\ \overline{POS}(n) &= \{p \in \mathbb{R}[x_1, \dots, x_n] : p(x_1, \dots, x_n) \geq 0, (x_1, \dots, x_n) \in \mathbb{R}^n\}, \\ SOS(n) &= \left\{ \sum_{i=1}^r p_i(x_1, \dots, x_n)^2 : p_i \in \mathbb{R}[x_1, \dots, x_n], 1 \leq i \leq r < \infty \right\}, \\ ELL(n) &= \{p \in \mathbb{R}[x_1, \dots, x_n] : p(x_1, \dots, x_n) \neq 0, (x_1, \dots, x_n) \in \mathbb{R}^n\}, \\ V_\alpha &= V \cap \mathbb{R}_\alpha[x_1, \dots, x_n], V \in \{POS(n), \overline{POS}(n), SOS(n), ELL(n)\}. \end{aligned}$$

These sets are the natural multivariate analogs of the ones defined in §1 for $n = 1$. Note that $SOS(n) \subseteq \overline{POS}(n)$ with strict inclusion in general. Let $\mathfrak{L}(n)$ be the space of all linear operators $T : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$. As is well-known (cf., e.g., [6]), any $T \in \mathfrak{L}(n)$ may be uniquely represented as an (*a priori* infinite order) linear partial differential operator with polynomial coefficients:

$$T = \sum_{\alpha \in \mathbb{N}_0^n} q_\alpha(\mathbf{x}) \partial^\alpha, \quad q_\alpha \in \mathbb{R}[x_1, \dots, x_n], \alpha \in \mathbb{N}_0^n, \quad (3.1)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\partial_{x_i} = \partial/\partial x_i$, $1 \leq i \leq n$. We then associate to T an n -parameter family of linear partial differential operators with *constant* coefficients $\{T_{\mathbf{y}}\}_{\mathbf{y} \in \mathbb{R}^n}$ given by

$$T_{\mathbf{y}} = \sum_{\alpha \in \mathbb{N}_0^n} q_\alpha(\mathbf{y}) \partial^\alpha$$

for any fixed $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. Moreover, for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we define an n -parameter family of polynomials $\{p_{\mathbf{y},\alpha}\}_{\mathbf{y} \in \mathbb{R}^n}$ by

$$p_{\mathbf{y},\alpha}(\mathbf{x}) = \sum_{\beta \leq \alpha} q_\beta(\mathbf{y}) \mathbf{x}^\beta \in \mathbb{R}[x_1, \dots, x_n]$$

for each fixed $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, where $\mathbf{x}^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ for $\mathbf{x} = (x_1, \dots, x_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, and we use the standard (product) partial order on \mathbb{N}_0^n : $\beta \leq \alpha$ provided that $\beta_i \leq \alpha_i$, $1 \leq i \leq n$. Let us also introduce an n -parameter family of real-valued maps $\{\mathcal{H}_{\mathbf{y}}\}_{\mathbf{y} \in \mathbb{R}^n}$ on the additive semigroup \mathbb{N}_0^n by setting

$$\mathcal{H}_{\mathbf{y}}(\alpha) = \alpha! q_\alpha(\mathbf{y}), \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n,$$

for any fixed $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, where as before $\alpha! = \alpha_1! \cdots \alpha_n!$.

Remark 3.1. Clearly, $p_{\mathbf{y},\alpha} \in \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_n]$ for each $\alpha \in \mathbb{N}_0^n$. By analogy with the terminology used in the univariate case (cf. Remark 2.1) we call this $2n$ variable polynomial the α -truncated symbol of the operator T .

Set $2\alpha = (2\alpha_1, \dots, 2\alpha_n) \in \mathbb{N}_0^n$ whenever $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. The following theorem characterizes all linear operators in $\mathfrak{L}(n)$ that preserve the cone $\overline{POS}(n)$ of non-negative polynomials in n variables.

Theorem 3.1. *For $T \in \mathfrak{L}(n)$ as in (3.1) the following assertions are equivalent:*

- (1) $T(\overline{POS}(n)) \subseteq \overline{POS}(n)$;
- (2) $T_{\mathbf{y}}(\overline{POS}(n)) \subseteq \overline{POS}(n)$ for all $\mathbf{y} \in \mathbb{R}^n$;
- (3) $\langle p_{\mathbf{y},2\alpha}, f \rangle \geq 0$ for all $\mathbf{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, $f \in \overline{POS}(n)_{2\alpha}$, i.e., $p_{\mathbf{y},2\alpha} \in \overline{POS}(n)_{2\alpha}^*$;
- (4) For any $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{R}^n$ there exists $\mu_{\mathbf{s}} \in \mathfrak{M}_n^+$ such that $\int \mathbf{t}^\alpha d\mu_{\mathbf{s}}(\mathbf{t}) \in \mathbb{R}[s_1, \dots, s_n]$, $\alpha \in \mathbb{N}_0^n$, and $\mathcal{H}_{\mathbf{y}}(\alpha) = \int \mathbf{t}^\alpha d\mu_{\mathbf{y}}(\mathbf{t})$, $\mathbf{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, where $\mathbf{t}^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$. The family of operators $\{T_{\mathbf{y}}\}_{\mathbf{y} \in \mathbb{R}^n}$ is then given by the convolutions

$$T_{\mathbf{y}}(f)(\mathbf{x}) = \int f(\mathbf{t} + \mathbf{x}) d\mu_{\mathbf{y}}(\mathbf{t}), \quad f \in \mathbb{R}[x_1, \dots, x_n], \mathbf{y} \in \mathbb{R}^n.$$

Furthermore, each of these conditions implies:

- (5) The map $\mathcal{H}_{\mathbf{y}} : \mathbb{N}_0^n \rightarrow \mathbb{R}$ is a positive semidefinite kernel on $\mathbb{N}_0^n \times \mathbb{N}_0^n$ for each $\mathbf{y} \in \mathbb{R}^n$, i.e., the Hankel matrix $(\mathcal{H}_{\mathbf{y}}(\alpha^i + \alpha^j))_{i,j=1}^m$ is positive semidefinite for any $m \in \mathbb{N}$ and $\{\alpha^1, \dots, \alpha^m\} \subset \mathbb{N}_0^n$.

Proof. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ the cone $\overline{POS}(n)_\alpha$ is invariant under shift operators $e^{\mathbf{a} \cdot \partial}$, $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, where $\mathbf{a} \cdot \partial = a_1 \partial / \partial x_1 + \cdots + a_n \partial / \partial x_n$. Moreover, one has

$$T_{\mathbf{y}}(f)(\mathbf{a}) = \sum_{\beta \leq \alpha} q_{\beta}(\mathbf{y}) \partial^\beta f(\mathbf{a}) = \langle p_{\mathbf{y},\alpha}, e^{\mathbf{a} \cdot \partial} f \rangle, \quad f \in \mathbb{R}_\alpha[x_1, \dots, x_n].$$

Note also that if $f \in \overline{POS}(n)$ then $\deg_{x_i}(f) \equiv 0 \pmod{2}$, $1 \leq i \leq n$. The equivalences (1) \Leftrightarrow (2) \Leftrightarrow (3) follow readily from these facts and straightforward extensions of the arguments in the proof of Theorem 2.1 to the multivariate case.

Next, since $T_{\mathbf{y}}(f)(\mathbf{0}) = \langle p_{\mathbf{y},\beta}, f \rangle$ if $f \in \mathbb{R}_\beta[x_1, \dots, x_n]$, $\beta \in \mathbb{N}_0^n$, Taylor's formula yields (4) \Rightarrow (3). To prove the converse, for $\mathbf{y} \in \mathbb{R}^n$ we let $L_{\mathbf{y}} : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}$ be the linear functional determined by $L_{\mathbf{y}}(\mathbf{x}^\beta) = \mathcal{H}_{\mathbf{y}}(\beta)$, $\beta \in \mathbb{N}_0^n$. It follows that

$$L_{\mathbf{y}}(f) = \langle p_{\mathbf{y},\beta}, f \rangle, \quad f \in \mathbb{R}_\beta[x_1, \dots, x_n], \beta \in \mathbb{N}_0^n.$$

In particular, if (3) holds then $L_{\mathbf{y}}(f) \geq 0$ for all $f \in \overline{POS}(n)_{2\alpha}$, $\alpha \in \mathbb{N}_0^n$. From Haviland's theorem (cf. §1) and the fact that $\mathcal{H}_{\mathbf{y}}(\beta)$ is a polynomial in \mathbf{y} for each $\beta \in \mathbb{N}_0^n$ we deduce that (3) \Rightarrow (4).

Let now $m \in \mathbb{N}$, $\{\alpha^1, \dots, \alpha^m\} \subset \mathbb{N}_0^n$, $c_i \in \mathbb{R}$, $1 \leq i \leq m$, and set

$$g(\mathbf{x}) = \sum_{i=1}^m c_i \mathbf{x}^{\alpha^i} \in \mathbb{R}[x_1, \dots, x_n].$$

If $\alpha \in \mathbb{N}_0^n$ is such that $\alpha^i + \alpha^j \leq \alpha$, $1 \leq i, j \leq m$, one has the identity

$$\langle p_{\mathbf{y},\alpha}, g^2 \rangle = \sum_{i,j=1}^m \mathcal{H}_{\mathbf{y}}(\alpha^i + \alpha^j) c_i c_j. \quad (3.2)$$

Since $g^2 \in \overline{POS}(n)$ this proves that (3) \Rightarrow (5). \square

The next theorem gives a characterization of all linear operators in $\mathfrak{L}(n)$ that preserve the cone $POS(n)$ of positive polynomials in n variables.

Theorem 3.2. *For $T \in \mathfrak{L}(n)$ as in (3.1) the following assertions are equivalent:*

- (1) $T(POS(n)) \subseteq POS(n)$;
- (2) $T_{\mathbf{y}}(POS(n)) \subseteq POS(n)$ for all $\mathbf{y} \in \mathbb{R}^n$;
- (3) $\langle p_{\mathbf{y}, 2\alpha}, f \rangle > 0$ for all $\mathbf{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, $f \in POS(n)_{2\alpha}$;
- (4) *For any $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{R}^n$ there exists $\mu_{\mathbf{s}} \in \mathfrak{M}_n^+$ whose support is not contained in a proper real algebraic variety² such that for all $\alpha \in \mathbb{N}_0^n$ one has $\int \mathbf{t}^\alpha d\mu_{\mathbf{s}}(\mathbf{t}) \in \mathbb{R}[s_1, \dots, s_n]$ and $\mathcal{H}_{\mathbf{y}}(\alpha) = \int \mathbf{t}^\alpha d\mu_{\mathbf{y}}(\mathbf{t})$, $\mathbf{y} \in \mathbb{R}^n$. The family of operators $\{T_{\mathbf{y}}\}_{\mathbf{y} \in \mathbb{R}^n}$ is then given by the convolutions*

$$T_{\mathbf{y}}(f)(\mathbf{x}) = \int f(\mathbf{t} + \mathbf{x}) d\mu_{\mathbf{y}}(\mathbf{t}), \quad f \in \mathbb{R}[x_1, \dots, x_n], \mathbf{y} \in \mathbb{R}^n.$$

Furthermore, each of these conditions implies:

- (5) *The map $\mathcal{H}_{\mathbf{y}} : \mathbb{N}_0^n \rightarrow \mathbb{R}$ is a positive definite kernel on $\mathbb{N}_0^n \times \mathbb{N}_0^n$ for each $\mathbf{y} \in \mathbb{R}^n$, i.e., the Hankel matrix $(\mathcal{H}_{\mathbf{y}}(\alpha^i + \alpha^j))_{i,j=1}^m$ is positive definite for any $m \in \mathbb{N}$ and $\{\alpha^1, \dots, \alpha^m\} \subset \mathbb{N}_0^n$.*

Proof. Since $POS(n)_{2\alpha}$ is invariant under shift operators $e^{\mathbf{a} \cdot \partial}$, $\mathbf{a} \in \mathbb{R}^n$, the arguments in the proof of Theorem 3.1 carry over *mutatis mutandis* to the present situation so we will not repeat them. The only novelty concerns the extra condition in (4) above on the support of the measures $\mu_{\mathbf{s}}$ (this condition does not appear in Theorem 3.1 (4)) for which we refer to e.g. Exercise 3.14 in [2, Chap. 6]. \square

Remark 3.2. Elementary homotopy arguments show that $ELL(n) = POS(n) \cup (-POS(n))$ and if $T \in \mathfrak{L}(n)$ then $T(ELL(n)) \subseteq ELL(n)$ if and only if either $T(POS(n)) \subseteq POS(n)$ or $T(POS(n)) \subseteq -POS(n)$. Therefore, Theorem 3.2 also yields a description of all linear operators in $\mathfrak{L}(n)$ preserving the set $ELL(n)$ of elliptic polynomials in n variables.

The above theorems solve the multivariate analogs of the “soft” (transcendental/unbounded degree) classification questions stated in Problem 2. It is clear that one gets “hard” (algebraic/bounded degree) theorems in similar fashion. More precisely, given $\alpha \in \mathbb{N}_0^n$ and a linear operator $T : \mathbb{R}_{2\alpha}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$ it follows from the proofs of Theorems 3.1–3.2 that

$$\begin{aligned} T(\overline{POS}(n)_{2\alpha}) \subseteq \overline{POS}(n) &\Leftrightarrow T_{\mathbf{y}}(\overline{POS}(n)_{2\alpha}) \subseteq \overline{POS}(n) \text{ for all } \mathbf{y} \in \mathbb{R}^n \\ &\Leftrightarrow \langle p_{\mathbf{y}, 2\alpha}, f \rangle \geq 0 \text{ for all } \mathbf{y} \in \mathbb{R}^n, f \in \overline{POS}(n)_{2\alpha} \end{aligned}$$

(i.e., $p_{\mathbf{y}, 2\alpha} \in \overline{POS}(n)_{2\alpha}^*$ for all $\mathbf{y} \in \mathbb{R}^n$) and

$$\begin{aligned} T(POS(n)_{2\alpha}) \subseteq POS(n) &\Leftrightarrow T_{\mathbf{y}}(POS(n)_{2\alpha}) \subseteq POS(n) \text{ for all } \mathbf{y} \in \mathbb{R}^n \\ &\Leftrightarrow \langle p_{\mathbf{y}, 2\alpha}, f \rangle > 0 \text{ for all } \mathbf{y} \in \mathbb{R}^n, f \in POS(n)_{2\alpha}. \end{aligned}$$

These equivalences answer the multivariate analogs of the classification questions raised in Problem 1.

4. RELATED RESULTS AND FURTHER DIRECTIONS

1. It is interesting to compare the classification theorems established here with those obtained in [3, 4, 6] for linear operators preserving real-rooted polynomials or their multivariate analogs. A common feature is the use of appropriately defined operator symbols, see Remarks 2.1, 2.3, and 3.1. However, the truncated symbols introduced here are employed in a different way than the symbols used in [3, 4]. Note for instance that if $T \in \mathfrak{L}(POS)$ is as in (2.1) then for each $m \in \mathbb{N}_0$ its $2m$ -truncated symbol $p_{y, 2m}(x)$ is a positive polynomial in variables x, y (since all even degree truncations of the exponential series are positive) but the converse statement

²In other words, $\text{supp}(\mu_{\mathbf{s}})$ is not a subset of $f^{-1}(0)$ for some $f \in \mathbb{R}[x_1, \dots, x_n] \setminus \{0\}$.

is not true, as one can see from e.g. Corollary 4.1 (b) below. By contrast, the main results in [4] essentially assert that a linear operator preserves the set of univariate real-rooted polynomials if and only if its symbol is a “real-rooted” polynomial in two variables (in some appropriate sense).

We should also note that there are plenty of linear differential operators of positive order in the Weyl algebra $\mathcal{A}_1(\mathbb{R})$ which preserve real-rootedness (these are characterized in [4, 6]) whereas no such operators preserve positive, non-negative or elliptic polynomials (cf. Corollary 4.1).

The equivalent conditions in our main results successively describe linear operators preserving non-negativity by means of linear differential operators with constant coefficients, Fischer-Fock dual cones, positive semidefinite Hankel forms, and convolution operators induced by non-negative measures (Theorems 2.1 and 3.1). The Fischer-Fock product was also used in Theorem 1.11 of [6] which asserts that an operator $T \in \mathcal{A}_1(\mathbb{R})$ preserves real-rootedness if and only if its Fischer-Fock adjoint T^* has the same property. Note that this duality fails for linear operators preserving positive, non-negative or elliptic polynomials since multiplication by a positive polynomial clearly preserves the sets SOS , POS , ELL but differential operators of positive order in $\mathcal{A}_1(\mathbb{R})$ do not. Nevertheless, Theorem 2.1 (3) and Theorem 3.1 (3) show that appropriate Fischer-Fock dualities (in one or several variables) do hold in the present context as well.

2. The classification theorems in §2.2–2.3 yield in particular the following improvements of the main results in [9] (Theorem A, Corollary, and Theorem B in *op. cit.*).

Corollary 4.1. *In the above notations one has:*

- (a) *If $T \in \mathfrak{L}$ is an element of the Weyl algebra $\mathcal{A}_1(\mathbb{R})$ of order $d \geq 1$ then for any even integer $\ell > d$ one has $T(V_\ell) \setminus V \neq \emptyset$ whenever $V \in \{POS, SOS, ELL\}$;*
- (b) *There are no linear ordinary differential operators with polynomial coefficients of finite positive order in $\mathfrak{L}(POS) \cup \mathfrak{L}(SOS) \cup \mathfrak{L}(ELL)$;*
- (c) *If $T \in \mathfrak{L} \setminus \{0\}$ is a linear ordinary differential operator with constant coefficients then $T \in \mathfrak{L}(POS) \Leftrightarrow T \in \mathfrak{L}(SOS) \Leftrightarrow T \in \mathfrak{L}(ELL) \Leftrightarrow T$ is a convolution operator of the form*

$$T(f)(x) = \int f(t+x) d\mu(t), \quad f \in \mathbb{R}[x],$$

for some $\mu \in \mathfrak{M}_1^+$.

Proof. To prove (a) it is enough to show that $T \notin \mathfrak{L}_\ell(V)$ for the smallest even integer $\ell > d$ and $V = SOS$ (the arguments for $V = POS$ and $V = ELL$ are similar). Let

$$T = \sum_{i=0}^d q_i(x) D^i, \quad q_i \in \mathbb{R}[x], \quad i \in \mathbb{N}_0, \quad q_d \neq 0,$$

and $y \in \mathbb{R}$ be such that $q_d(y) \neq 0$. If d is odd set $q_{d+1}(y) = 0$. Then the determinant of the lower right 2×2 principal submatrix of $\mathcal{H}_{y,d+1}$ is $-q_d(y)^2$ so $\mathcal{H}_{y,d+1}$ is not positive semidefinite and by Theorem 2.3 $T \notin \mathfrak{L}_{d+1}(SOS)$. If d is even set $q_{d+1}(y) = q_{d+2}(y) = 0$ and suppose that $T \in \mathfrak{L}_d(SOS)$ (if $T \notin \mathfrak{L}_d(SOS)$ there is nothing to prove since $SOS_d \subset SOS_{d+2}$). By Theorem 2.3 $\mathcal{H}_{y,d}$ is positive semidefinite hence $q_d(y) > 0$. The determinant of the lower right 3×3 principal submatrix of $\mathcal{H}_{y,d+2}$ equals $-q_d(y)^3$ and thus $\mathcal{H}_{y,d+2}$ cannot be positive semidefinite. By Theorem 2.3 again we conclude that $T \notin \mathfrak{L}_{d+2}(SOS)$.

Clearly, (a) \Rightarrow (b) while (c) follows from Theorem 2.1 and the fact that if $T \in \mathfrak{L}(SOS)$ and $T(1) \equiv 0$ then $T \equiv 0$ (cf. §2.1). \square

Remark 4.1. Corollary 4.1 (a) asserts that a linear ordinary differential operator of order $d \geq 1$ with polynomial coefficients cannot preserve positivity, non-negativity, or ellipticity when acting on $\mathbb{R}_\ell[x]$, where ℓ is any even integer greater than d . In particular, this implies part (b) stating that there are no linear ordinary differential operators of finite positive order (acting on $\mathbb{R}[x]$) that preserve positivity, non-negativity, or ellipticity. Finally, part (c) asserts that a non-trivial linear ordinary differential operator with constant coefficients (acting on $\mathbb{R}[x]$) preserves each of the sets POS , SOS , ELL provided that it preserves at least one of them and any such operator is a convolution with a non-negative measure with all finite moments.

3. From Theorems 3.1–3.2 we deduce that any linear partial differential operator with constant coefficients that preserves non-negativity or positivity is actually a convolution operator induced by a non-negative measure with all finite moments. More precisely, we have the following multivariate analog of Corollary 4.1 (c).

Corollary 4.2. *Let $T \in \mathfrak{L}(n)$ be a linear partial differential operator with constant coefficients.*

- (i) *The following assertions are equivalent:*
 - (a) $T(\overline{POS}(n)) \subseteq \overline{POS}(n)$;
 - (b) *There exists $\mu \in \mathfrak{M}_n^+$ such that T may be represented as the convolution*

$$T(f)(\mathbf{x}) = \int f(\mathbf{t} + \mathbf{x}) d\mu(\mathbf{t}), \quad f \in \mathbb{R}[x_1, \dots, x_n].$$

- (ii) *The following assertions are also equivalent:*
 - (c) $T(POS(n)) \subseteq POS(n)$;
 - (d) *There exists $\mu \in \mathfrak{M}_n^+$ whose support is not contained in a proper real algebraic variety such that T may be represented as the convolution*

$$T(f)(\mathbf{x}) = \int f(\mathbf{t} + \mathbf{x}) d\mu(\mathbf{t}), \quad f \in \mathbb{R}[x_1, \dots, x_n].$$

A similar corollary follows for diagonal operators in the standard monomial basis of $\mathbb{R}[x_1, \dots, x_n]$, i.e., operators $T \in \mathfrak{L}(n)$ given by $T(\mathbf{x}^\beta) = \lambda_\beta \mathbf{x}^\beta$, where $\lambda_\beta \in \mathbb{R}$, $\beta \in \mathbb{N}_0^n$. It is easy to see that such an operator may be uniquely represented as

$$T = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \mathbf{x}^\alpha \partial^\alpha, \quad a_\alpha \in \mathbb{R}, \alpha \in \mathbb{N}_0^n,$$

and that $\lambda_\beta = \sum_{\alpha \leq \beta} (\beta)_\alpha a_\alpha$ for $\beta \in \mathbb{N}_0^n$, where as usual $(\beta)_\alpha = (\beta_1)_{\alpha_1} \cdots (\beta_n)_{\alpha_n}$ and $(m)_k = m(m-1) \cdots (m-k+1)$ whenever $k, m \in \mathbb{N}_0$ with $k \leq m$. Elementary computations then yield the following implication:

$$\alpha! a_\alpha = \int \mathbf{t}^\alpha d\mu(\mathbf{t}), \alpha \in \mathbb{N}_0^n \implies \lambda_\beta = \int (\mathbf{t} + \mathbf{1})^\beta d\mu(\mathbf{t}) = \int \mathbf{s}^\beta d\nu(\mathbf{s}), \beta \in \mathbb{N}_0^n,$$

where $\mathbf{1}$ denotes the all ones vector and $\mathbf{s} = \mathbf{t} + \mathbf{1}$ (note in particular that for μ, ν as above one has $\mu \in \mathfrak{M}_n^+ \Leftrightarrow \nu \in \mathfrak{M}_n^+$). Hence, if the condition in the left-hand side of the above implication holds we deduce that the diagonal operator T is given by

$$T(f)(\mathbf{x}) = \int f(\mathbf{s}\mathbf{x}) d\nu(\mathbf{s}), \quad f \in \mathbb{R}[x_1, \dots, x_n],$$

where $\mathbf{s}\mathbf{x} = (s_1 x_1, \dots, s_n x_n)$. The next statement is therefore an immediate consequence of these arguments and Theorems 3.1–3.2.

Corollary 4.3. *Let $T \in \mathfrak{L}(n)$ be a diagonal operator given by $T(\mathbf{x}^\beta) = \lambda_\beta \mathbf{x}^\beta$, $\lambda_\beta \in \mathbb{R}$, $\beta \in \mathbb{N}_0^n$.*

- (i) *The following assertions are equivalent:*
 - (a) $T(\overline{POS}(n)) \subseteq \overline{POS}(n)$;

- (b) *There exists $\nu \in \mathfrak{M}_n^+$ such that T may be represented as*

$$T(f)(\mathbf{x}) = \int f(\mathbf{s}\mathbf{x})d\nu(\mathbf{s}), \quad f \in \mathbb{R}[x_1, \dots, x_n].$$

- (ii) *The following assertions are also equivalent:*

- (c) $T(POS(n)) \subseteq POS(n)$;
 (d) *There exists $\nu \in \mathfrak{M}_n^+$ whose support is not contained in a proper real algebraic variety such that T may be represented as*

$$T(f)(\mathbf{x}) = \int f(\mathbf{s}\mathbf{x})d\nu(\mathbf{s}), \quad f \in \mathbb{R}[x_1, \dots, x_n].$$

Remark 4.2. Let $T \in \mathfrak{L}(n)$ be either a partial differential operator with constant coefficients or a diagonal operator. Note that $T(1)$ is then a constant polynomial and suppose that $T(\overline{POS}(n)) \subseteq \overline{POS}(n)$. If $T(1) = 0$ it follows from Corollary 4.2 (i) (respectively, Corollary 4.3 (i)) that the non-negative measure μ (respectively, ν) has total mass zero and then $T \equiv 0$. Therefore, if $T \neq 0$ then $T(1) > 0$ and so if $f \in POS(n)$ one gets $T(f) = T(f - a) + aT(1) \in POS(n)$, where $a > 0$ is such that $f - a \in POS(n)$ (clearly, such a exists for any $f \in POS(n)$). We conclude that if $T \in \mathfrak{L}(n) \setminus \{0\}$ is as in Corollary 4.2 or Corollary 4.3 then $T(\overline{POS}(n)) \subseteq \overline{POS}(n) \Leftrightarrow T(POS(n)) \subseteq POS(n)$.

4. If $n \geq 2$ one has in general $SOS(n) \subsetneq \overline{POS}(n)$ and it is natural to ask for a characterization of the set $\{T \in \mathfrak{L}(n) : T(SOS(n)) \subseteq \overline{POS}(n)\}$ (obviously, this set contains the semigroup of all $T \in \mathfrak{L}(n)$ such that $T(\overline{POS}(n)) \subseteq \overline{POS}(n)$ that we already described in Theorem 3.1). The following theorem provides an answer to this question.

Theorem 4.4. *For $n \geq 2$ and $T \in \mathfrak{L}(n)$ as in (3.1) the following are equivalent:*

- (1) $T(SOS(n)) \subseteq \overline{POS}(n)$;
- (2) $T_{\mathbf{y}}(SOS(n)) \subseteq \overline{POS}(n)$ for all $\mathbf{y} \in \mathbb{R}^n$;
- (3) $\langle p_{\mathbf{y}, 2\alpha}, f \rangle \geq 0$ for all $\mathbf{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, $f \in SOS(n)_{2\alpha}$, i.e., $p_{\mathbf{y}, 2\alpha} \in SOS(n)_{2\alpha}^*$;
- (4) *The map $\mathcal{H}_{\mathbf{y}} : \mathbb{N}_0^n \rightarrow \mathbb{R}$ is a positive semidefinite kernel on $\mathbb{N}_0^n \times \mathbb{N}_0^n$ for each $\mathbf{y} \in \mathbb{R}^n$, i.e., the Hankel matrix $(\mathcal{H}_{\mathbf{y}}(\alpha^i + \alpha^j))_{i,j=1}^m$ is positive semidefinite for any $m \in \mathbb{N}$ and $\{\alpha^1, \dots, \alpha^m\} \subset \mathbb{N}_0^n$.*

Proof. Since $SOS(n) = \cup_{\alpha \in \mathbb{N}_0^n} SOS(n)_{2\alpha}$ and each cone $SOS(n)_{2\alpha}$ is invariant under shift operators $e^{\mathbf{a} \cdot \partial}$, $\mathbf{a} \in \mathbb{R}^n$, the implications (1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) follow as in the proof of Theorem 3.1. To show that (4) \Rightarrow (3) note that $f \in SOS(n)_{2\alpha}$ if and only if $f = \sum_{j=1}^r g_j^2$ for some $r \in \mathbb{N}$ and $g_j \in \mathbb{R}_{\alpha}[x_1, \dots, x_n]$, $1 \leq j \leq r$. Given $\alpha \in \mathbb{N}_0^n$ let $m = m(\alpha)$ be the cardinality of $\{\beta \in \mathbb{N}_0^n : \beta \leq \alpha\}$ (which is clearly finite) and order this set lexicographically as $\{\alpha^1, \dots, \alpha^m\}$. Any $g \in \mathbb{R}_{\alpha}[x_1, \dots, x_n]$ may then be written as

$$g(\mathbf{x}) = \sum_{i=1}^m c_i \mathbf{x}^{\alpha^i}, \quad c_i \in \mathbb{R}, \quad 1 \leq i \leq m,$$

so the desired conclusion follows from (3.2). \square

In view of Theorem 4.4 and Corollary 4.2 one may ask whether the following analog of Hilbert's 17th problem holds, cf. [7, Problem 17].

Question. *Is it true that for any $p \in \overline{POS}(n)$ there exist $f \in SOS(n)$ and a linear partial differential operator with constant coefficients $T \in \mathfrak{L}(n)$ such that $T(SOS(n)) \subseteq \overline{POS}(n)$ and $T(f) = p$?*

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